

Double-horizon limit and decoupling of the dynamics at the horizon

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ABSTRACT: We show that the *attractor mechanism* for generic black hole is a consequence of the double-horizon. Investigation of equations of motion shows that in the case of the double-horizon black holes, the dynamics of the geometry, the scalars and the gauge fields at the horizon decouples from the rest of the space. In the general case, the value of the fields at the horizon satisfies a number of differential equations of functions of the θ coordinate. We show this for the case of rotating and non-rotating electrically charged black holes in the general two derivative theories of gravity and $f(R)$ gravities including the theories with cosmological constant.

KEYWORDS: Black Holes, Black Holes in String Theory.

JHEP01(2007)060

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1. Introduction

The attractor mechanism for black holes states that the value of the scalar fields (moduli fields) at the horizon, in certain class of black holes (extremal black holes), is fixed and is independent of the large distance boundary conditions [1].

It was first shown for black holes in supersymmetric theories but recent careful analysis has confirmed it for non-supersymmetric case [2]–[4]. This generalization is studied by several works from different points of view. [5]–[10].

Almost at the same time of this progress, Sen has developed his entropy function method [11]–[12]. In this method the $AdS_2 \times S^2$ geometry of the near horizon geometry of the spherically symmetric extremal black holes plays the central role and the parameters of the near horizon are fixed by extremizing a function called the *entropy function*. Charges of the black hole play main role in fixing the parameters of the near horizon geometry and including the values of the moduli. Therefore all the relevant information about horizon is fixed independent of the asymptotic behavior. Following this progress, a number of articles have appeared in studying various aspects of entropy function method [13]–[23]. It is also suggested in [4] that physical reason for the attractor mechanism is in existence of a *throat geometry* near the horizon of an extremal black hole. This throat causes that spatial distance between an arbitrary point near the horizon to the horizon of an extremal black hole diverges. In the non-extremal case there is no such throat and this distance is finite. In [4] it is argued that this infinite distance makes the information of the spatial infinity be lost when one approaches the horizon. Hence due to this infinite separation the parameters of the horizon become independent of the asymptotic values. The power of the entropy function method is in using this throat geometry in the definition of the extremal black hole. In [24], existence of the throat for the extremal rotating black holes is shown,

therefor it seems that the throat geometry is a property of all known extremal black holes leading to the attractor mechanism.

In the proof of the attractor mechanism for non-supersymmetric spherically symmetric charged black holes in [4], the double-horizon property of the extremal black holes is used to reduce the equations of motion for scalar fields to equations which gives the value of scalars at the horizon without any need to asymptotic information, more than this, they have shown that the infinite throat of an extremal black hole is a consequence of the double-horizon condition. Therefore it seems logical that imposing the double-horizon conditions on the equations of motion leads to the same results of the entropy function method does. In this work we investigate this point.

It is shown that imposing the double-horizon condition plus certain regularity of the physical properties at the horizon converts equations of motion at the horizon to a system of equations which decouple from the bulk. This decoupling occurs as all r-derivative terms in the equations of motions disappear at the horizon. Role of double-horizon becomes completely obvious. For distinct horizons this decoupling is not achieved and dynamics of the fields at the horizon, including the scalar moduli do not decouple from the bulk.

We show our statement for both rotating and non-rotating electrically charged black holes for general two-derivative theories of gravity which couple to a number of scalars and U(1) gauge fields.

For non-rotating case e.o.m at the horizon are converted to a system of algebraic equations which can be solved without using any other information of the bulk. The final equations which determine field configuration of the horizon are very similar to equation which come from entropy function method.

For rotating black holes equation at the horizon are not algebraic, they are differential equations. Like non-rotating case the equations are decoupled from the bulk. Solutions to these equations need boundary conditions. These boundary conditions are values of the fields at the poles of the horizon. Necessity of this boundary values makes it difficult to immediately deduce the attractor mechanism for rotating black holes.

Adding higher order $f(R)$ corrections does not change the decoupling of the horizon from the bulk for double-horizon black holes. This point extends our result to this class of theories.

2. Decoupling of dynamics at the horizon and the double-horizon condition

In this section we explore the role of the double-horizon condition in the decoupling of the dynamics of the fields at the horizon from the rest of the space.

We consider a theory of gravity with scalars and U(1) gauge fields. For simplicity the theory with only one scalar and one gauge field is studied. Generalizations to theories with more scalar and gauge fields are straightforward. The action which describes this theory is given by

$$S = \frac{1}{\kappa^2} \int d^4x \sqrt{-G} (R - 2\partial_\mu \Phi \partial^\mu \Phi - f(\Phi) F_{\mu\nu} F^{\mu\nu}) \quad (2.1)$$

After studying this model we show that our analysis will work for more complicated actions which involves different functions of R in particular R^n gravity and theories with cosmological constant.

Equations of motion are

$$R_{\mu\nu} - 2\partial_\mu\Phi\partial_\nu\Phi = f(\Phi)\left(2F_{\mu\lambda}F_\nu{}^\lambda - \frac{1}{2}G_{\mu\nu}F_{\kappa\lambda}F^{\kappa\lambda}\right) \quad (2.2)$$

$$\frac{1}{\sqrt{-G}}\partial_\mu(\sqrt{-G}\partial^\mu\Phi) = \frac{1}{4}\frac{\delta f(\Phi)}{\delta\Phi}F_{\mu\nu}F^{\mu\nu} \quad (2.3)$$

$$\partial_\mu(\sqrt{-G}f(\Phi)F^{\mu\nu}) = 0 \quad (2.4)$$

The ansatz we consider in the following subsection describes different black holes of this theory, including the electrically charged, both rotating and non rotating .

2.1 Non-rotating electrically charged black holes

The ansatz for the fields of a non-rotating black hole with only electrical charge is

$$ds^2 = -\frac{(r-r_+)(r-r_-)}{A(r)}dt^2 + \frac{A(r)}{(r-r_+)(r-r_-)}dr^2 + B(r)(d\theta^2 + \sin^2(\theta)d\phi^2) \quad (2.5)$$

$$F_{rt} = e(r) \quad \Phi = \Phi(r) \quad (2.6)$$

where r_+ and r_- are the radii of the outer and the inner horizons. $A(r)$, $B(r)$ and $e(r)$ the radial component of the electric field, are non-zero regular functions. $\Phi(r)$ the scalar field is also finite at the horizon. For brevity we show the value of these functions at the horizon by A , B , e and Φ . This form of the metric, scalar and gauge field are consequences of the spherical symmetry.

On the horizon we have

$$F \equiv F_{\mu\nu}F^{\mu\nu} \Big|_{r=r_+} = -2e^2 \quad (2.7)$$

Allowing the two horizons merge in the equations of motion and using (2.5)–(2.7), and (2.3) on the outer horizon we obtain,

$$\frac{1}{A}(r_+ - r_-)\left(\frac{\partial}{\partial r}\Phi(r)\right) \Big|_{r=r_+} = -\frac{1}{2}e^2\frac{\delta f}{\delta\Phi} \Big|_{r=r_+} \quad (2.8)$$

The left hand side of this equation vanishes when we apply the double-horizon assumption i.e. $r_+ = r_-$ and finiteness of Φ ,

$$\frac{\delta f}{\delta\Phi} \Big|_{r=r_+} = 0 \quad (2.9)$$

Whatever the behavior of the fields at the infinity, the value of the scalar field must satisfy this equation which can be solved without any other information related to the bulk. It is even independent of the charge of the black hole. This is an attractor equation for

a non rotating charged black hole in which the role of double horizon in removing the r-derivative of $\Phi(r)$ and hence independence from the Φ in the bulk of the space in particular far distance is obvious.

This equation is in agreement with [2]. It is the simplified form of the equation

$$\frac{\delta V_{\text{eff}}}{\delta \Phi} = 0 \quad (2.10)$$

that has been used in [2] to obtain the value of Φ on the horizon. However, there is a difference, in [2], (2.10) has been imposed as a condition for attractor mechanism but we have derived it by imposing double-horizon condition on the e.o.m. Therefore double-horizon condition forces this equation and is a sufficient condition for the attractor mechanism. Our further analysis shows its necessity.

Next we consider the effect of the double-horizon on the other equations. If we write equation (2.2) at the horizon, the double-horizon condition converts it to

$$f(\Phi)Ae^2 = 1 \quad (2.11)$$

$$f(\Phi)Be^2 = 1 \quad (2.12)$$

Using (2.5) and (2.6) to simplify the equation (2.4), we obtain ;

$$\frac{d}{dr} \left(\sin\theta B(r) f(\Phi) e(r) \right) = 0 \quad (2.13)$$

The double-horizon condition does not have any effect on this equation and thus there is no mechanism to remove the derivative with respect to radial directions. However, this equation introduces a conserved quantity which for our case is the charge of the black hole. By integrating both sides of this equation we get

$$\frac{d}{dr} \left(\int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta B(r) f(\Phi) e(r) \right) = 0 \quad (2.14)$$

Hence

$$\int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta B(r) f(\Phi) e(r) = \pi Q \quad (2.15)$$

where Q is the charge of the black hole. This equation does not have any dependence on radial coordinate, thus at the horizon we have

$$Bf(\Phi)e = \frac{1}{4}Q \quad (2.16)$$

Equations (2.9), (2.11), (2.12) and (2.16) are attractor equations. They completely determine all the information at the horizon without any reference to the information in the bulk.

Entropy of this black hole is equal to $\frac{A_r}{4}$ where A_r is the area of the horizon. Using (2.12) and (2.16) , entropy is given by

$$\varepsilon = \frac{\pi}{16} \frac{Q^2}{f(\Phi)} \quad (2.17)$$

As we see, the double-horizon condition is just sufficient to remove the r-derivative terms in the equations which originate from the variation of metric and scalar field. However the equation of the gauge field is not decoupled by double-horizon condition. But the charge of the black hole is sufficient to solve this equation and specify the value of the electromagnetic fields at the horizon. Therefore by knowing the charge of the double-horizon black holes we can extract every information about the horizon. If these equations have unique solutions then it results a *generalized attractor mechanism* which states that the value of the fields on the horizon of a double-horizon black hole is independent of the asymptotic values. For the case where the equations have more than one solution the conclusion is not immediate like [2] where we must consider dynamics of the fields near the horizon to determine which of these solutions is reached. In this case the black hole may assume different phases corresponding to the different values of the scalar moduli.

2.2 Rotating electrically charged black holes

We consider the same theory which is described by the action (2.1). Equations of motion are (2.2)–(2.4). We shall investigate a rotating black hole in this theory. We take the angular momentum of this black hole to be along the z direction. Our ansatz for the metric of the rotating black hole is

$$ds^2 = B(r, \theta) \left(- \frac{S(r)}{A(r) - a^2 \sin^2 \theta S(r)} dt^2 + \frac{1}{S(r)} dr^2 + d\theta^2 \right) + \sin^2(\theta) \frac{A(r) - a^2 \sin^2 \theta S(r)}{B(r, \theta)} \left(d\phi - \frac{C(r) - E(r, \theta) S(r)}{A(r) - a^2 \sin^2 \theta S(r)} dt \right)^2 \quad (2.18)$$

where $a = \frac{J}{M}$, J is the angular momentum, M is the mass of the black hole and we have,

$$S(r) \equiv (r - r_+)(r - r_-) \quad (2.19)$$

where r_+ and r_- are respectively radii of the outer and the inner horizons and functions $A(r)$, $B(r, \theta)$, $C(r)$ and $E(r, \theta)$ are non-zero regular functions at the horizon. We show their value at the outer horizon by A , $B(\theta)$, C and $E(\theta)$ which are regular function at the horizon.

Unlike the non-rotating case the scalar field Φ depends on r and θ coordinates:

$$\Phi = \Phi(r, \theta) \quad (2.20)$$

Independence from ϕ -coordinate results from azimuthal symmetry.

It is assumed that our black hole is only electrically charged. Our ansatz for the gauge field is

$$A_t = b(r, \theta) \quad A_\phi = -a \sin^2 \theta b(r, \theta) \quad (2.21)$$

Therefore only F_{rt} , $F_{r\phi}$, $F_{\theta t}$ and $F_{\theta\phi}$ of $F_{\mu\nu}$ tensor are non-zero:

$$F_{rt} = \frac{\partial}{\partial r} A_t \equiv e(r, \theta) \quad F_{r\phi} = -a \sin^2 \theta e(r, \theta) \quad (2.22)$$

$$F_{\theta t} = \frac{\partial}{\partial \theta} b(r, \theta) \quad F_{\theta\phi} = \frac{\partial}{\partial \theta} (-a \sin^2 \theta b(r, \theta)) \quad (2.23)$$

We show the value of $b(r, \theta)$ and $e(r, \theta)$ at the horizon by $b(\theta)$ and $e(\theta)$.

Our ansatz for the field configuration of a rotating black hole is compatible with the Kerr-Newman solution and the solutions of [25].¹

By using symmetry consideration we can deduce that at the horizon

$$B(\theta = 0) = B(\theta = \pi) \quad (2.24)$$

$$\Phi(\theta = 0) = \Phi(\theta = \pi) \quad (2.25)$$

$$b(\theta = 0) = b(\theta = \pi) \quad (2.26)$$

As a physical condition we demand that all the physical quantities to be finite on the horizon. Using (2.18), (2.22) and (2.23) we get

$$\begin{aligned} F_{\mu\nu}F^{\mu\nu} = & -\frac{2}{B(r, \theta)} \left(\frac{A(r) - a^2 \sin^2\theta S(r)}{B(r, \theta)} F_{rt}^2 + \frac{C(r)^2}{B(r, \theta)(A(r) - a^2 \sin^2\theta S(r))} F_{r\phi}^2 \right. \\ & \left. + 2 \frac{C(r) - S(r)E(r, \theta)}{B(r, \theta)} F_{rt}F_{r\phi} \right) \\ & -\frac{2}{B(r, \theta)S(r)} \left(\frac{A(r) - a^2 \sin^2\theta S(r)}{B(r, \theta)} F_{\theta t}^2 + \frac{C(r)^2}{B(r, \theta)(A(r) - a^2 \sin^2\theta S(r))} F_{\theta\phi}^2 \right. \\ & \left. + 2 \frac{C(r) - S(r)E(r, \theta)}{B(r, \theta)} F_{\theta t}F_{\theta\phi} \right) \\ & + \frac{2}{B(r, \theta)} \left(S(r)F_{r\phi}^2 + F_{\theta\phi}^2 \right) \left(\frac{B(r, \theta)^2 - 2\sin^2\theta C(r)E(r, \theta)}{\sin^2\theta B(r, \theta)(A(r) - a^2 \sin^2\theta S(r))} \right. \\ & \left. - \frac{S(r)E(r, \theta)^2}{B(r, \theta)(A(r) - a^2 \sin^2\theta S(r))} \right) \end{aligned} \quad (2.27)$$

Finiteness of $F_{\mu\nu}F^{\mu\nu}$ at the horizon requires that

$$F_{\theta t} + \frac{C}{A}F_{\theta\phi} = 0 \quad (2.28)$$

This is a reduced form for the following equation:

$$F_{\theta t} + \frac{C(r) - E(r, \theta)S(r)}{A(r) - a^2 \sin^2\theta S(r)} F_{\theta\phi} = V(r, \theta)S(r) \quad (2.29)$$

where $V(r, \theta)$ is a non-zero finite function at the horizon.

An immediate result of this equation is that

$$F \equiv F_{\mu\nu}F^{\mu\nu} \Big|_{r=r_+} = \frac{2}{A \sin^2\theta} \left(1 + \frac{a^2 \sin^4\theta C^2}{A B(\theta)^2} \right) F_{\theta\phi}^2 - \frac{2A}{B(\theta)^2} \left(F_{rt} + \frac{C}{A}F_{r\phi} \right)^2 \quad (2.30)$$

¹Solutions of [25] are completely general and consist of black holes of more than one charge but for the case that black hole has only one kind of charge, our ansatz is compatible with those solutions.

Using (2.29) and imposing the double-horizon condition we have

$$\alpha \equiv \frac{\partial}{\partial r} \frac{C(r)}{A(r)} \Big|_{r=r_h} = - \frac{\partial}{\partial r} \left(\frac{F_{\theta t}}{F_{\theta\phi}} \right) \Big|_{r=r_h} \quad (2.31)$$

where α is a constant. We can calculate it by using only the information at the horizon, because

$$\frac{\partial}{\partial r} \frac{F_{\theta t}}{F_{\theta\phi}} = \frac{1}{F_{\theta\phi}^2} \left(F_{\theta\phi} \frac{\partial}{\partial r} \frac{\partial}{\partial \theta} (A_t) - F_{\theta t} \frac{\partial}{\partial r} \frac{\partial}{\partial \theta} (A_\phi) \right) \quad (2.32)$$

by exchanging order of $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ and using (2.28) we get

$$\begin{aligned} \alpha &= - \frac{1}{F_{\theta\phi}^2} \left(F_{\theta\phi} \frac{d}{d\theta} F_{rt} - F_{\theta t} \frac{d}{d\theta} F_{r\phi} \right) \\ &= - \frac{1}{F_{\theta\phi}} \frac{d}{d\theta} \left(F_{rt} + \frac{C}{A} F_{r\phi} \right) \end{aligned} \quad (2.33)$$

For Kerr-Newman solution α is given by

$$\alpha = \frac{-2J}{(M^2 + a^2)^2} \quad (2.34)$$

After deriving the consequences of the finiteness of the physical quantities at the horizon, we consider the role of the double-horizon condition on the form of the e.o.m at the horizon. Our claim is that, this condition converts the equations at the horizon to a system of equations which are completely independent of the radial coordinate.

We start from the equation (2.3). At the outer horizon it reads;

$$\frac{1}{\sqrt{-G}} \frac{\partial}{\partial r} \left(\sqrt{-G} \frac{S(r)}{B(r, \theta)} \frac{\partial \Phi}{\partial r} \right) \Big|_{r=r_+} + \frac{1}{\sqrt{-G}} \frac{\partial}{\partial \theta} \left(\sqrt{-G} \frac{1}{B(r, \theta)} \frac{\partial \Phi}{\partial \theta} \right) \Big|_{r=r_+} = \frac{1}{4} \frac{\delta f}{\delta \Phi} F \quad (2.35)$$

Since $S(r)$ and $\frac{d}{dr} S(r)$ are zero at the horizon the double-horizon condition forces the first term of the l.h.s of this equation vanish. Thus we obtain

$$\frac{1}{\sqrt{-G}} \frac{d}{d\theta} \left(\sqrt{-G} \frac{1}{B(\theta)} \frac{d\Phi}{d\theta} \right) = \frac{1}{4} \frac{\delta f}{\delta \Phi} F \quad (2.36)$$

whereupon using (2.18), gives

$$G = -B(\theta)^2 \sin^2 \theta. \quad (2.37)$$

Substituting (2.37) and (2.30) in (2.36) gives the following equation:

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Phi}{d\theta} \right) = \frac{1}{2} \frac{\delta f}{\delta \Phi} \left(\frac{B(\theta)}{A \sin^2 \theta} \left(1 + \frac{a^2 \sin^4 \theta C^2}{A B(\theta)^2} \right) F_{\theta\phi}^2 - \frac{A}{B(\theta)} \left(F_{rt} + \frac{C}{A} F_{r\phi} \right)^2 \right) \quad (2.38)$$

which can be written as

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Phi}{d\theta} \right) = \frac{\delta V_{\text{eff}}}{\delta \Phi} \quad (2.39)$$

with V_{eff} defined to be

$$V_{\text{eff}} = \frac{1}{2}f(\Phi)\frac{A}{B(\theta)}\left(\left(\frac{B(\theta)}{A\sin\theta}F_{\theta\phi}\right)^2\left(1 + \frac{a^2\sin^4\theta C^2}{AB(\theta)^2}\right) - \left(F_{rt} + \frac{C}{A}F_{r\phi}\right)^2\right) \quad (2.40)$$

We see again that the double-horizon condition is responsible for the removal of the r-derivative. This is one of the attractor equations but unlike the non-rotating case where the scalar decouples from the other fields, determination of scalar field at the horizon relies on the knowledge of the metric and gauge field strength albeit restricted to the horizon. Concluding the attractor mechanism for the scalar field needs the demonstration of the independence of the geometry and the gauge field strength at the horizon from the asymptotic value of the scalar field. We show this point by investigating the other equations of motion.

Another important difference with the non-rotating case is that (2.39) is a differential equation, thus complete determination of scalar field at the horizon needs imposing certain boundary conditions. These boundary conditions are only at the horizon but if some attractor mechanism is at work it is required that the boundary conditions be independent of the asymptotic values. Proof of this point needs more considerations.

Now we consider imposing the double-horizon condition on the gravity part of the equations at the horizon. A straightforward calculation using the metric (2.18) shows that the non-zero components of the Ricci tensor are R_{rr} , $R_{r\theta}$, $R_{\theta\theta}$, R_{tt} , $R_{\phi\phi}$ and $R_{t\phi}$.

At the horizon, the equation (2.2) for the $R_{rr}R_{tt}$, $R_{\phi\phi}$ and $R_{t\phi}$ components reduces to

$$1 + \frac{1}{2}\frac{d}{d\theta}\left(\frac{1}{B(\theta)}\frac{d}{d\theta}B(\theta)\right) + \frac{\cot\theta}{2B(\theta)}\frac{d}{d\theta}B(\theta) - \frac{1}{2}\alpha^2\frac{A^2}{B(\theta)^2}\sin^2\theta = -\frac{A}{B(\theta)}f(\Phi)H(\theta) \quad (2.41)$$

where α is given by (2.33) and

$$H(\theta) = \left(\left(F_{rt} + \frac{C}{A}F_{r\phi}\right)^2 + \left(1 - \frac{a^2\sin^4\theta C^2}{AB(\theta)^2}\right)\left(\frac{B(\theta)F_{\theta\phi}}{A\sin\theta}\right)^2\right) \quad (2.42)$$

For the $R_{\theta\theta}$, the equation (2.2) at the horizon simplifies and takes the form,

$$-1 + \frac{1}{2B(\theta)}\frac{d^2}{d\theta^2}B(\theta) - \frac{3\cot\theta}{2B(\theta)}\frac{d}{d\theta}B(\theta) = 2\left(\frac{d}{d\theta}\Phi(\theta)\right)^2 + f(\Phi)\frac{A}{B(\theta)}H(\theta) \quad (2.43)$$

For the $R_{r\theta}$ component, equation (2.2) at the horizon reads as

$$F_{rt}F_{\theta t} + \frac{C^2}{A^2}F_{r\phi}F_{\theta\phi} + \frac{C}{A}(F_{rt}F_{\theta\phi} + F_{r\phi}F_{\theta t}) = 0 \quad (2.44)$$

But if we use (2.28), this equation is clearly satisfied. It is obvious that again the double-horizon condition has removed all the r-derivative terms in the above equations.

Now we consider the equation (2.4) from which we obtain;

$$\frac{\partial}{\partial r}\left(\sqrt{-G}f(\Phi)F^{rt}\right) + \frac{\partial}{\partial\theta}\left(\sqrt{-G}f(\Phi)F^{\theta t}\right) = 0 \quad (2.45)$$

$$\frac{\partial}{\partial r}\left(\sqrt{-G}f(\Phi)F^{r\phi}\right) + \frac{\partial}{\partial\theta}\left(\sqrt{-G}f(\Phi)F^{\theta\phi}\right) = 0 \quad (2.46)$$

Using (2.18), we can write

$$F^{r\phi} = \frac{C(r) - E(r, \theta)S(r)}{A(r) - a^2 \sin^2\theta S(r)} F^{rt} + \frac{S(r)}{\sin^2\theta A(r)} F_{r\phi} \quad (2.47)$$

$$F^{\theta\phi} = \frac{C(r) - E(r, \theta)S(r)}{A(r) - a^2 \sin^2\theta S(r)} F^{\theta t} + \frac{1}{\sin^2\theta A(r)} F_{\theta\phi} \quad (2.48)$$

Substituting (2.47) and (2.48) in (2.46) and imposing the double-horizon assumption, at the horizon we find;

$$\begin{aligned} & \frac{C}{A} \left(\frac{\partial}{\partial r} (\sqrt{-G} f(\Phi) F^{rt}) \right) \Big|_{r=r_+} + \frac{\partial}{\partial \theta} (\sqrt{-G} f(\Phi) F^{\theta t}) \Big|_{\theta=0} + \\ & + (\sqrt{-G} f(\Phi) F^{rt}) \frac{\partial}{\partial r} \left(\frac{C(r)}{A(r)} \right) \Big|_{r=r_+} + \frac{1}{A} \frac{d}{d\theta} \left(\frac{1}{\sin^2(\theta)} \sqrt{-G} f(\Phi) F_{\theta\phi} \right) = 0 \end{aligned} \quad (2.49)$$

Using (2.45) and (2.31) in (2.49), we can remove the r-derivatives to obtain,

$$\alpha (\sqrt{-G} f(\Phi) F^{rt}) + \frac{1}{A} \frac{d}{d\theta} \left(\frac{1}{\sin^2(\theta)} \sqrt{-G} f(\Phi) F_{\theta\phi} \right) = 0 \quad (2.50)$$

Finally substitution of (2.37) and (2.33) in this equation gives;

$$f(\Phi)^2 \frac{d}{d\theta} \left(F_{rt} + \frac{C}{A} F_{r\phi} \right)^2 + \frac{d}{d\theta} \left(\frac{f(\Phi) B(\theta)}{A \sin\theta} F_{\theta\phi} \right)^2 = 0 \quad (2.51)$$

Integrating the equation (2.45) results;

$$\frac{\partial}{\partial r} \left(\int_0^\pi d\theta \sqrt{-G} f(\Phi) F^{rt} \right) + \left[\sqrt{-G} f(\Phi) F^{\theta t} \right]_{\theta=0}^{\theta=\pi} = 0. \quad (2.52)$$

The second term vanishes due to $\sin\theta$ coefficient in $\sqrt{-G}$ and the regularity of the components of $F_{\mu\nu}$ at the horizon. Hence the first term introduces a constant quantity Q , the charge of the black hole:

$$\int_0^\pi d\theta \sqrt{-G} f(\Phi) F^{rt} = \frac{1}{2} Q \quad (2.53)$$

Integrating (2.50) and substituting in this equation implies that:

$$\frac{1}{2} \alpha Q = -\frac{1}{A} \left[\frac{\sqrt{-G} f(\Phi) F_{\theta\phi}}{\sin^2\theta} \right]_{\theta=0}^{\theta=\pi} \quad (2.54)$$

Use of (2.33) and (2.37) in this equation gives;

$$\frac{d}{d\theta} \left(F_{rt} + \frac{C}{A} F_{r\phi} \right) - \frac{\beta}{A} F_{\theta\phi} = 0 \quad (2.55)$$

where

$$\beta = \left[\frac{2f(\Phi) B(\theta) F_{\theta\phi}}{Q \sin\theta} \right]_{\theta=0}^{\theta=\pi} \quad (2.56)$$

which simplifies with the aid of (2.23)–(2.26) give

$$\beta = \frac{8a}{Q} f(\Phi) B(\theta) b(\theta) \Big|_{\theta=0} \quad (2.57)$$

β is a constant and depends on the values of the fields at the poles of the horizon. Values of the fields at the poles are the necessary boundary conditions in our discussion, thus knowing this value will determine β . It is notable that β depends on the parameters of the black hole, for example for Kerr-Newman solution we have

$$\beta = 2J \quad (2.58)$$

Using (2.33) and (2.55) we deduce,

$$\alpha = -\frac{\beta}{A} \quad (2.59)$$

Equations (2.28), (2.39), (2.41), (2.43), (2.51) and (2.55) are a complete set of independent differential equations which determine A , $B(\theta)$, C , $\Phi(\theta)$, $b(\theta)$ and $e(\theta)$ (we summarize them at the appendix). These equations are independent of the radial direction and the complete solution of them only needs values of the fields at the poles of the horizon as the boundary condition. These equations include the constant α , β , a and Q which depend on the physical observable of the solution. Only two quantities are independent.

Independence of the dynamics of the fields at the horizon from the bulk is a consequence of the double horizon condition but the final conclusion on the independence of the values of the fields at the horizon from asymptotic values (attractor mechanism) needs more considerations, first we must investigate the independence of the boundary conditions (values of the fields at the poles of the horizon) from the asymptotic values and second, the solution must be unique. When there are, more than one solution like non-rotating case we must consider dynamics of the fields near the horizon. It is enough to perform the analysis near one of the poles.

3. Generalization to f(R) gravities

The essential role of the double-horizon condition in isolating equations of motion at the horizon from the bulk also occurs in the theories of gravity with higher order corrections.

A simple generalization of the theory considered so far, is addition of cosmological constant. The action has the form

$$S = \frac{1}{\kappa^2} \int d^4x \sqrt{-G} (R - 2\Lambda - 2\partial_\mu \Phi \partial^\mu \Phi - f(\Phi) F_{\mu\nu} F^{\mu\nu}) \quad (3.1)$$

where Λ is the cosmological constant.

The equation of motion from variation of scalar and gauge field are unchanged and they are the same as (2.3) and (2.4). The equation of motion from variation of metric is now given by,

$$R_{\mu\nu} - \Lambda G_{\mu\nu} - 2\partial_\mu \Phi \partial_\nu \Phi = f(\Phi) \left(2F_{\mu\lambda} F_\nu{}^\lambda - \frac{1}{2} G_{\mu\nu} F_{\kappa\lambda} F^{\kappa\lambda} \right) \quad (3.2)$$

It is not difficult to see that by taking the ansatz (2.18), (2.20) and (2.21) for the fields of the black hole solutions of this theory, we only have to slightly modify the equation (2.41) and (2.43) by solely adding the term $\Lambda B(\theta)$. Independence of the dynamics of the fields at the horizon from the bulk in this case is then clear.

Another example is the R^n gravities. The action is given by

$$S = \frac{1}{\kappa^2} \int d^4x \sqrt{-G} (R + \gamma R^n - 2\partial_\mu \Phi \partial^\mu \Phi - f(\Phi) F_{\mu\nu} F^{\mu\nu}) \quad (3.3)$$

The equations of motion from the variation of metric are,

$$R_{\mu\nu} + \gamma R^{n-1} (2R_{\mu\nu} - \frac{n-1}{2} R G_{\mu\nu}) - 2\partial_\mu \Phi \partial_\nu \Phi = f(\Phi) \left(2F_{\mu\lambda} F_\nu{}^\lambda - \frac{1}{2} G_{\mu\nu} F_{\kappa\lambda} F^{\kappa\lambda} \right) \quad (3.4)$$

The equations of motion from the variation of the scalar and gauge field are unchanged and the same as (2.3) and (2.4).

The extra term in the equation (3.4) adds the following term to the equations (2.41):

$$\gamma I_1(\theta)^{n-1} \left(I_2(\theta) - \frac{A^2 \sin^2 \theta}{B(\theta)^2} \alpha^2 + \frac{n-1}{2} I_1(\theta) B(\theta) \right) \quad (3.5)$$

where

$$I_1(\theta) = -\frac{A^2 \sin^2 \theta}{2B(\theta)^3} \alpha^2 + \frac{1}{2B(\theta)} \frac{d}{d\theta} \left(\frac{1}{B(\theta)} \frac{d}{d\theta} B(\theta) \right) + \frac{1}{B(\theta)^2} \frac{d^2}{d\theta^2} B(\theta) - \frac{\cot \theta}{B(\theta)^2} \frac{d}{d\theta} B(\theta) \quad (3.6)$$

$$I_2(\theta) = 2 + \frac{d}{d\theta} \left(\frac{1}{B(\theta)} \frac{d}{d\theta} B(\theta) \right) + \frac{\cot \theta}{B(\theta)} \frac{d}{d\theta} B(\theta) \quad (3.7)$$

For the equation (2.43) the additional term is

$$\gamma I_1(\theta)^{n-1} \left(I_3(\theta) - \frac{n-1}{2} I_1(\theta) B(\theta) \right) \quad (3.8)$$

where

$$I_3(\theta) = -2 + \frac{1}{B(\theta)} \frac{d^2}{d\theta^2} B(\theta) - \frac{3\cot \theta}{B(\theta)} \frac{d}{d\theta} B(\theta) \quad (3.9)$$

Other equations are unchanged. It is clear that our claim remains valid and there is no r-derivative terms in the equations of motion. Hence the dynamics of the fields at the horizon is independent of the bulk. This shows that for every well-defined function of R which is expandable in terms of the powers of R , this decoupling occurs. Therefore the double-horizon condition is a necessary condition for attractor mechanism in $f(R)$ gravities and is responsible for the decoupling of the dynamics at the horizon from the rest of the space.

4. Conclusion

Our analysis shows that the double-horizon condition in the theories we have considered, namely $f(R)$ theories, has capability of decoupling the dynamics at the horizon of the black hole solutions. This isolation is done by removal of the radial coordinate derivatives in the equations of motion. For non-rotating black holes, e.o.m at the horizon are completely

algebraic and it is possible to solve them without any other information. If the solution is unique, it leads to the attractor mechanism, if not, the resolution needs detail considerations. One must analyze the stability of the solution with respect to changes in the bulk.

For the rotating black hole, e.o.m are differential equations of functions of θ coordinate. Solving them needs boundary conditions. These boundary conditions are values of the fields at the poles of the black hole. Necessity of these boundary condition makes it difficult to immediately arrive at the conclusion about the attractor mechanism. The attractor mechanism needs the investigation of the independence of the value of the fields at the poles from asymptotic values. Since the equations we are considering are on a compact manifold (sphere) and are non-linear, single valuedness may act as boundary condition. Therefore it is plausible that consistency condition will dictate a certain class of boundary condition(s) on the solutions. Moreover if the solution to equations of motion at the horizon is not unique stability analysis is required to differentiate between different configurations. Again detailed analysis is needed like what has been done for non-rotating case in [2]. In any case solutions to the set of the equations obtained, specify the only possible field configurations and will limit the moduli at the horizon. Any field configuration must assume one of these forms as its limit at the horizon. Such questions are under investigation by the authors.

We have considered black holes with only electric charge but by changing our ansatz for the gauge fields, it is possible to investigate black holes with magnetic charge too. It is possible to generalize this method for black holes with more than one kind of charge and the theories of gravity with more complicated higher order corrections.

Our method clearly shows why the attractor mechanism does not exist for black holes with distinct horizons. In these cases the dynamics of the fields at the outer horizon are not independent of the bulk and so it is not possible to isolate the horizon from the rest of the space. All known extremal black holes have double-horizon, thus our analysis includes this class of black holes. However, finding clear relation between extremity and double-horizon limit needs further investigations.

If we accept the double-horizon condition as the necessary condition for the attractor mechanism, role of supersymmetry will be negligible and one can conclude that it is not the supersymmetry which fixes the moduli at the horizon. Fixing moduli is a consequence of infinite throat which makes the time to reach the horizon from any point in the bulk to diverge and become infinite. Our analysis supports this point for generic black holes by direct investigation of the dynamics of the fields at the horizon.

Note added

During the time that we were preparing this paper, the attractor mechanism has been investigated for rotating black holes by entropy function method [26]. Our results are in completely agreement with their results for the case of two-derivative black holes which they have studied in that paper.

Acknowledgments

We would like to thank F.Ardalan, M. M. Sheikh Jabbari and M. Alishahiha for their useful comments and discussions. We would also like to thank the Iranian chapter of TWAS for partial support.

A. Attractor equations for charged rotating black holes

We collect the equations of the section (2.2) in this section:

- Metric of the rotating black hole:

$$ds^2 = B(r, \theta) \left(- \frac{S(r)}{A(r) - a^2 \sin^2 \theta S(r)} dt^2 + \frac{1}{S(r)} dr^2 + d\theta^2 \right) + \sin^2(\theta) \frac{A(r) - a^2 \sin^2 \theta S(r)}{B(r, \theta)} \left(d\phi - \frac{C(r) - E(r, \theta) S(r)}{A(r) - a^2 \sin^2 \theta S(r)} dt \right)^2 \quad (\text{A.1})$$

- Condition for the finiteness of the $F_{\mu\nu} F^{\mu\nu}$ at the horizon:

$$F_{\theta t} + \frac{C}{A} F_{\theta\phi} = 0 \quad (\text{A.2})$$

- Equations of motion at the horizon after imposing double horizon condition:

$$\frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{d\Phi}{d\theta} \right) = \frac{\delta V_{\text{eff}}}{\delta \Phi} \quad (\text{A.3})$$

$$1 + \frac{1}{2} \frac{d}{d\theta} \left(\frac{1}{B(\theta)} \frac{d}{d\theta} B(\theta) \right) + \frac{\cot\theta}{2B(\theta)} \frac{d}{d\theta} B(\theta) - \frac{1}{2} \alpha^2 \frac{A^2}{B(\theta)^2} \sin^2\theta = -\frac{A}{B(\theta)} f(\Phi) H(\theta) \quad (\text{A.4})$$

$$-1 + \frac{1}{2B(\theta)} \frac{d^2}{d\theta^2} B(\theta) - \frac{3\cot\theta}{2B(\theta)} \frac{d}{d\theta} B(\theta) = 2 \left(\frac{d}{d\theta} \Phi(\theta) \right)^2 + f(\Phi) \frac{A}{B(\theta)} H(\theta) \quad (\text{A.5})$$

$$F_{rt} F_{\theta t} + \frac{C^2}{A^2} F_{r\phi} F_{\theta\phi} + \frac{C}{A} (F_{rt} F_{\theta\phi} + F_{r\phi} F_{\theta t}) = 0 \quad (\text{A.6})$$

$$f(\Phi)^2 \frac{d}{d\theta} \left(F_{rt} + \frac{C}{A} F_{r\phi} \right)^2 + \frac{d}{d\theta} \left(\frac{f(\Phi) B(\theta)}{A \sin\theta} F_{\theta\phi} \right)^2 = 0 \quad (\text{A.7})$$

$$\frac{d}{d\theta} \left(F_{rt} + \frac{C}{A} F_{r\phi} \right) - \frac{\beta}{A} F_{\theta\phi} = 0 \quad (\text{A.8})$$

where

$$V_{\text{eff}} = \frac{1}{2} f(\Phi) \frac{A}{B(\theta)} \left(\left(\frac{B(\theta)}{A \sin\theta} F_{\theta\phi} \right)^2 \left(1 + \frac{a^2 \sin^4 \theta C^2}{A B(\theta)^2} \right) - \left(F_{rt} + \frac{C}{A} F_{r\phi} \right)^2 \right) \quad (\text{A.9})$$

$$H(\theta) = \left(\left(F_{rt} + \frac{C}{A} F_{r\phi} \right)^2 + \left(1 - \frac{a^2 \sin^4 \theta C^2}{A B(\theta)^2} \right) \left(\frac{B(\theta) F_{\theta\phi}}{A \sin\theta} \right)^2 \right) \quad (\text{A.10})$$

$$\beta = \frac{8a}{Q} f(\Phi) B(\theta) b(\theta) \Big|_{\theta=0} \quad (\text{A.11})$$

$$\alpha = -\frac{\beta}{A} \quad (\text{A.12})$$

Note that these equations are independent of our ansatz for the gauge field.

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